

Gravitomagnetic bending angle of light with finite-distance corrections in stationary axisymmetric spacetimes

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Abstract

By using the Gauss-Bonnet theorem, the bending angle of light in a static, spherically symmetric and asymptotically flat spacetime has been recently discussed, especially by taking account of the finite distance from a lens object to a light source and a receiver [Ishihara, Suzuki, Ono, Asada, Phys. Rev. D **95**, 044017 (2017)]. We discuss a possible extension of the method of calculating the bending angle of light to stationary, axisymmetric and asymptotically flat spacetimes. For this purpose, we consider the light rays on the equatorial plane in the axisymmetric spacetime. We introduce a spatial metric to define the bending angle of light in the finite-distance situation. We show that the proposed bending angle of light is coordinate-invariant by using the Gauss-Bonnet theorem. The non-vanishing geodesic curvature of the photon orbit with the spatial metric is caused in gravitomagnetism, even though the light ray in the four-dimensional spacetime follows the null geodesic. Finally, we consider Kerr spacetime as an example in order to examine how the bending angle of light is computed by the present method. The finite-distance correction to the gravitomagnetic deflection angle due to the Sun's spin is around a pico-arcsecond level. The finite-distance corrections for Sgr A* also are estimated to be very small. Therefore, the gravitomagnetic finite-distance corrections for these objects are unlikely to be observed with present technology.

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I. INTRODUCTION

In 1919 [1], the gravitational bending of light by the Sun led to one of the earliest experimental confirmations of the theory of general relativity [2]. Since then, the gravitational bending of light not only by the Sun but also by other astronomical objects has been observed at many times. The gravitational lensing is one of the most useful tools in modern astronomy and cosmology.

The gravitational bending of light is an important subject also in the theoretical study of gravity, for example on a null structure of a spacetime. Hagihara found the analytical solution for null geodesics with the Schwarzschild metric [3], where the expressions involve incomplete elliptic integrals of the first kind. See [4] for exact solutions for light trajectories in Schwarzschild and some other black hole solutions. See e.g. [5, 6] for a review on the light deflection in the weak-field approximation of Schwarzschild spacetime. A generalized lens equation for the light deflection in Schwarzschild metric in the weak-field approximation and valid for finite distances of source and observer from the lens was discussed by Zschocke [7].

Since the pioneering work by Darwin [8], moreover, the strong deflection limit has attracted a lot of interests (See e.g. [9–12]), mainly because we expect that the recent progress in astronomical instruments will soon enable us to detect such strong deflection phenomena. For better understanding of a strong gravitational field, a more precise formulation of the bending angle was vastly studied [13–20]; Strong-field gravitational lensing in a Schwarzschild black hole was investigated by Frittelli, Kling and Newman [13], by Virbhadra and Ellis [14] and more comprehensively by Virbhadra [15]; distinctive lensing features of wormholes and naked singularities were studied by many authors [16–25]. Kitamura, Nakajima and Asada proposed a unified model for the Schwarzschild lens and the Ellis wormhole lens, as a one parameter family in the inverse powers of the distance ($1/r^n$) [26], in which some models might follow a non-standard equation of state [12, 27–29]. See also Tsukamoto et al. (2015) [30] for a possible connection between this inverse power model and the Tangherlini solution to the higher-dimensional Einstein equation.

Gibbons and Werner (2008) proposed an alternative way of deriving the deflection angle of light [31], in which they assumed that the source and receiver are located at an asymptotic region and they used the Gauss-Bonnet theorem in differential geometry to a spatial domain described by the optical metric that allows us to describe a light ray as a spatial curve. By

extending Gibbons and Werner’s idea, Ishihara et al. have recently investigated finite-distance corrections in the small deflection case (corresponding to a large impact parameter case) [32] and also in the strong deflection limit for which the photon orbits may have the winding number larger than unity [33]. In particular, the asymptotic receiver and source have not been assumed.

However, the earlier treatments [32, 33] are limited within the spherical symmetry. It is not clear whether the Gauss-Bonnet method with using the optical metric can be extended to axisymmetric cases or not. This is mostly because there can exist off-diagonal (time-space) components of the spacetime metric in an axisymmetric spacetime. The time-space components seem to make it unclear whether the optical metric can be constructed. After the gravitational lensing by a spinning object [34–36] and that by a relativistic binary [37] were discussed extensively by perturbative approaches such as the post-Newtonian approximation, Werner (2012) [38] proposed the use of the Kerr-Randers optical geometry on this issue [39]. To be more precise, he used the osculating Riemann approach in Finsler geometry in order to discuss the lensing by the Kerr black hole, for which the metric can be written in the Randers form. However, this approach requires that the endpoints (namely, the source and the receiver) of the photon orbit are in Euclidean space, for which angles can be easily defined. This requirement is mainly because jump angles at the vertices in the Gauss-Bonnet theorem are problematic in the Finsler geometry. Namely, it is unlikely that the Finsler geometry can be used for computing the finite-distance corrections.

Therefore, the main purpose of the present paper is to extend the earlier formulation in Refs. [32, 33], especially in order to examine finite-distance corrections to the deflection angle of light in the axisymmetric spacetime, for which the gravitational deflection of light may include gravitomagnetic effects (e.g. [34–37]). The geometrical setups in the present paper are not those in the optical geometry, in the sense that the photon orbit has a non-vanishing geodesic curvature, though the light ray in the four-dimensional spacetime obeys a null geodesic.

This paper is organized as follows. Section II briefly summarizes the formulation for the gravitational deflection angle of light by using the Gauss-Bonnet theorem by following Refs. [32, 33], where the spacetime is assumed to be spherically symmetric. Section III discusses a possible extension to stationary and axisymmetric spacetimes. In particular, it is shown that the proposed definition of the deflection angle is coordinate-invariant by using the Gauss-

Bonnet theorem. Section IV uses the Kerr metric as a known example of the stationary and axisymmetric spacetimes in order to discuss how to compute the gravitational deflection angle of light by the proposed method. Section V is devoted to conclusion.

Throughout this paper, we use the unit of $G = c = 1$, and the observer may be called the receiver in order to avoid a confusion between r_O and r_0 by using r_R .

II. GRAVITATIONAL DEFLECTION OF LIGHT IN STATIC, SPHERICALLY SYMMETRIC SPACETIMES

A. Optical metric

This section briefly summarizes the alternative approach developed by Ishihara et al. [32, 33] to the gravitational deflection of light, where they assume static, spherically symmetric and asymptotically flat spacetimes. These spacetimes can be described as (cf. Eq.(23.3) in [5])

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2, \end{aligned} \tag{1}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. This metric form allows us to consider a wormhole solution with a throat as well as a black hole spacetime. If we choose $C(r) = r^2$, then, r denotes the circumference radius.

Without the loss of generality, henceforth, we choose the photon orbital plane as the equatorial plane ($\theta = \pi/2$). Light rays satisfy the null condition as $ds^2 = 0$, which is rearranged as, via Eq. (1),

$$\begin{aligned} dt^2 &= \gamma_{IJ} dx^I dx^J \\ &= \frac{B(r)}{A(r)} dr^2 + \frac{C(r)}{A(r)} d\phi^2, \end{aligned} \tag{2}$$

where I and J denote 1, 2 and 2. γ_{IJ} is called the optical metric.

The optical metric on the equatorial plane, namely γ_{IJ} with $I, J = 1, 2$ for $\theta = \pi/2$, defines a two-dimensional Riemannian space (denoted as M^{opt}), in which the light ray is a spatial geodesic curve.

B. Orbit equation and angles at the source and receiver

Let us define the impact parameter of the light ray as

$$\begin{aligned} b &\equiv \frac{L}{E} \\ &= \frac{C(r)}{A(r)} \frac{d\phi}{dt}, \end{aligned} \quad (3)$$

such that the orbit equation can be obtained as

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{C(r)}{B(r)} = \frac{[C(r)]^2}{b^2 A(r) B(r)}. \quad (4)$$

In M^{opt} , let Ψ denote the angle of the light ray measured from the radial direction. We obtain

$$\cos \Psi = \frac{b\sqrt{A(r)B(r)}}{C(r)} \frac{dr}{d\phi}, \quad (5)$$

and

$$\sin \Psi = \frac{b\sqrt{A(r)}}{\sqrt{C(r)}}, \quad (6)$$

where we used Eq. (4).

Let Ψ_R and Ψ_S denote the angles that are measured at the receiver position (R) and the source position (S), respectively. Let $\phi_{RS} \equiv \int_S^R d\phi$ denote the coordinate separation angle between the receiver and source. We follow Ref. [32, 33] to define

$$\alpha \equiv \Psi_R - \Psi_S + \phi_{RS} \quad (7)$$

from the three angles Ψ_R , Ψ_S and ϕ_{RS} . This definition requires a comparison between angles at distinct points R and S . Moreover, it seems to depend on a choice of the coordinate ϕ .

However, Eq. (7) is geometrically invariant, particularly coordinate-independent. This is shown not only in the small deflection approximation [32] but even in the strong deflection limit [33]. A key for the proof is the Gauss-Bonnet theorem: Suppose that T is a two-dimensional orientable surface with boundaries ∂T_a ($a = 1, 2, \dots, N$) that are differentiable curves. See Figure 1. Let the jump angles between the curves be θ_a ($a = 1, 2, \dots, N$). Then, the Gauss-Bonnet theorem can be expressed as [42]

$$\iint_T K dS + \sum_{a=1}^N \int_{\partial T_a} \kappa_g d\ell + \sum_{a=1}^N \theta_a = 2\pi, \quad (8)$$

where K denotes the Gaussian curvature of the surface T , dS is the area element of the surface, κ_g means the geodesic curvature of ∂T_a , and ℓ is the line element along the boundary. The sign of the line element is chosen such that it is compatible with the orientation of the surface.

Let us consider a quadrilateral ${}^\infty_R\Box_S^\infty$, which consists of the spatial curve for the light ray, two outgoing radial lines from R and from S and a circular arc segment C_r of coordinate radius r_C ($r_C \rightarrow \infty$) centered at the lens which intersects the radial lines through the receiver or the source. See Figure 2 for the configuration such as the domain ${}^\infty_R\Box_S^\infty$. See also Ref. [33] for the case that the winding number is larger than unity. Henceforth, we restrict ourselves within the asymptotically flat spacetime, for which $\kappa_g \rightarrow 1/r_C$ and $d\ell \rightarrow r_C d\phi$ as $r_C \rightarrow \infty$ (See e.g. [31]). Hence, $\int_{C_r} \kappa_g d\ell \rightarrow \phi_{RS}$. Applying this result to the Gauss-Bonnet theorem for ${}^\infty_R\Box_S^\infty$, we obtain

$$\begin{aligned}\alpha &= \Psi_R - \Psi_S + \phi_{RS} \\ &= - \iint_{{}^\infty_R\Box_S^\infty} K dS.\end{aligned}\tag{9}$$

This shows that Eq. (7) is coordinate-invariant. Moreover, it follows that $\alpha = 0$ in Euclidean space.

As a practical way of computing the deflection angle of light, it is worthwhile to rewrite Eq. (7) as

$$\alpha = \int_{u_R}^{u_0} \frac{du}{\sqrt{F(u)}} + \int_{u_S}^{u_0} \frac{du}{\sqrt{F(u)}} + \Psi_R - \Psi_S.\tag{10}$$

Here, u is the inverse of r , u_0 means the inverse of the closest approach (often denoted as r_0), u_R and u_S denote the inverse of r_R and r_S , respectively, and $F(u)$ is defined as

$$F(u) \equiv \left(\frac{du}{d\phi} \right)^2,\tag{11}$$

which can be computed from Eq. (4). It follows that Eq. (10) recovers the deflection angle of light in the far limit of the source and the receiver ($r_S, r_R \rightarrow \infty$) as

$$\alpha_\infty = 2 \int_0^{u_0} \frac{du}{\sqrt{F(u)}} - \pi.\tag{12}$$

III. EXTENSION TO AXISYMMETRIC SPACETIMES

Henceforth, we assume a stationary and axisymmetric spacetime, for which we shall define the gravitational deflection angle of light by using the Gauss-Bonnet theorem.

A. Stationary, axisymmetric spacetime and optical metric

We consider a stationary axisymmetric spacetime. The line element for this spacetime is [43–45]

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -A(y^p, y^q) dt^2 - 2H(y^p, y^q) dt d\phi \\ &\quad + F(y^p, y^q) (\gamma_{pq} dy^p dy^q) + D(y^p, y^q) d\phi^2, \end{aligned} \quad (13)$$

where μ, ν run from 0 to 3, p, q take 1 and 2, t and ϕ coordinates are associated with the Killing vectors, and γ_{pq} is a two-dimensional symmetric tensor. It is more convenient to reexpress this metric into a form in which γ_{pq} is diagonalized. The present paper prefers the polar coordinates rather than the cylindrical ones, because the Kerr metric in the polar coordinates is considered in Section IV. In the polar coordinates, Eq. (13) becomes [46]

$$\begin{aligned} ds^2 &= -A(r, \theta) dt^2 - 2H(r, \theta) dt d\phi \\ &\quad + B(r, \theta) dr^2 + C(r, \theta) d\theta^2 + D(r, \theta) d\phi^2. \end{aligned} \quad (14)$$

The null condition $ds^2 = 0$ is solved for dt as [36]

$$dt = \sqrt{\gamma_{ij} dx^i dx^j} + \beta_i dx^i, \quad (15)$$

where i, j run from 1 to 3, γ_{ij} and β_i are defined as

$$\gamma_{ij} dx^i dx^j \equiv \frac{B(r, \theta)}{A(r, \theta)} dr^2 + \frac{C(r, \theta)}{A(r, \theta)} d\theta^2 + \frac{A(r, \theta)D(r, \theta) + H^2(r, \theta)}{A^2(r, \theta)} d\phi^2, \quad (16)$$

$$\beta_i dx^i \equiv -\frac{H(r, \theta)}{A(r, \theta)} d\phi. \quad (17)$$

This spatial metric $\gamma_{ij} (\neq g_{ij})$ may define the arc length (ℓ) along the light ray as

$$d\ell^2 \equiv \gamma_{ij} dx^i dx^j, \quad (18)$$

for which γ^{ij} is defined by $\gamma^{ij}\gamma_{jk} = \delta^i_k$. Note that ℓ defined in this way is an affine parameter along the light ray. See e.g. Appendix of Ref. [36] for the proof on the affine parameter [47].

γ_{ij} defines a 3-dimensional Riemannian space ${}^{(3)}M$ in which the motion of the photon is described as a motion in a spatial curve. The unit tangential vector along the spatial curve is defined as

$$e^i \equiv \frac{dx^i}{d\ell}. \quad (19)$$

The light ray follows the Fermat's principle [20]. By using the variational principle, this gives the equation for the light ray as [36]

$$e^i{}_{|k}e^k = a^i, \quad (20)$$

where $|$ denotes the covariant derivative with γ_{ij} and a^i is defined as

$$a^i \equiv \gamma^{ij}(\beta_{k|j} - \beta_{j|k})e^k. \quad (21)$$

Here,

$$e^i{}_{|k}e^k = \frac{de^i}{d\ell} + {}^{(3)}\Gamma^i{}_{jk}e^je^k, \quad (22)$$

where ${}^{(3)}\Gamma^i{}_{jk}$ denotes the Christoffel symbol associated with γ_{ij} .

The vector a^i is the spatial vector that means the acceleration originated from β_i . In particular, a^i is caused in gravitomagnetism as discussed below in more detail. This has an analogy as the acceleration by the Lorentz force $\propto \vec{v} \times (\vec{\nabla} \times \vec{A}_m)$ in electromagnetism, where \vec{A}_m denotes the magnetic vector potential.

We should note that γ_{ij} is not an induced metric. As a result, the photon orbit can deviate from a geodesic in ${}^{(3)}M$ with γ_{ij} if $\beta_i \neq 0$, even though the light ray in the four-dimensional spacetime follows the null geodesic.

For a stationary and spherically symmetric spacetime, one can always find a set of suitable coordinates such that g_{0i} can vanish to lead to $a^i = 0$. In this case, the photon orbit becomes a spatial geodesic curve in ${}^{(3)}M$.

The present paper discusses an extension to axisymmetric cases, which allow $g_{0i} \neq 0$. Therefore, we have to take account of non-zero κ_g along the photon orbit in the Gauss-Bonnet theorem. This non-vanishing κ_g of the photon orbit makes a crucial difference from the previous papers [32, 33]

B. Geodesic curvature and equatorial plane

Let us imagine a parameterized curve in a surface. The geodesic curvature of the parameterized curve is the surface-tangential component of acceleration (namely curvature) of the curve, while the normal curvature is the surface-normal component. The normal curvature

has nothing to do with the present paper. The geodesic curvature can be defined in the vector form as (e.g. [48])

$$\kappa_g \equiv \vec{T}' \cdot (\vec{T} \times \vec{N}), \quad (23)$$

where we assume a parameterized curve with a parameter, \vec{T} is the unit tangent vector for the curve by reparameterizing the curve using its arc length, \vec{T}' is its derivative with respect to the parameter, and \vec{N} is the unit normal vector for the surface. In this paper, Eq. (23) can be rewritten in the tensor form as

$$\kappa_g = \epsilon_{ijk} N^i a^j e^k, \quad (24)$$

where \vec{T} and \vec{T}' correspond to e^k and a^j , respectively. Here, the Levi-Civita tensor ϵ_{ijk} is defined by $\epsilon_{ijk} \equiv \sqrt{\gamma} \varepsilon_{ijk}$, where $\gamma \equiv \det(\gamma_{ij})$, and ε_{ijk} is the Levi-Civita symbol ($\varepsilon_{123} = 1$). In the present paper, the space is $^{(3)}M$. Therefore, we use γ_{ij} in the above definitions but not g_{ij} .

For a case of $a^i \neq 0$ due to g_{0i} , there can exist a non-vanishing integral of the geodesic curvature along the light ray in the Gauss-Bonnet theorem by Eq. (8).

By substituting Eq. (21) into a^i in Eq. (24), we obtain

$$\kappa_g = -\epsilon^{ijk} N_i \beta_{j|k}, \quad (25)$$

where we use $\gamma_{ij} e^i e^j = 1$.

Up to this point, the surface in $^{(3)}M$ is not specified. Henceforth, we focus on the equatorial motion of the photon. We choose $\theta = \pi/2$ as the equatorial plane. Then, the unit normal vector for the equatorial plane can be expressed as

$$N_p = \frac{1}{\sqrt{\gamma_{\theta\theta}}} \delta_p^\theta, \quad (26)$$

where we choose the upward direction without loss of generality.

For the equatorial case, one can show

$$\epsilon^{\theta pq} \beta_{q|p} = -\frac{1}{\sqrt{\gamma}} \beta_{\phi,r}, \quad (27)$$

where the comma denotes the partial derivative, we use $\epsilon^{\theta r \phi} = -1/\sqrt{\gamma}$ and we note $\beta_{r,\phi} = 0$ owing to the axisymmetry. By using Eqs. (26) and (27), an explicit form of κ_g in Eq. (25) is obtained as

$$\kappa_g = -\frac{1}{\sqrt{\gamma \gamma_{\theta\theta}}} \beta_{\phi,r}. \quad (28)$$

C. Impact parameter and the photon directions at the receiver and source

We study the orbit equation on the equatorial plane with Eq. (14). Associated with the two Killing vectors, there are the two constants of motion as

$$E = A(r)\dot{t} + H(r)\dot{\phi}, \quad (29)$$

$$L = D(r)\dot{\phi} - H(r)\dot{t}, \quad (30)$$

where the dot denotes the derivative with respect to the affine parameter.

As usual, we define the impact parameter as

$$\begin{aligned} b &\equiv \frac{L}{E} \\ &= \frac{-H(r)\dot{t} + D(r)\dot{\phi}}{A(r)\dot{t} + H(r)\dot{\phi}} \\ &= \frac{-H(r) + D(r)\frac{d\phi}{dt}}{A(r) + H(r)\frac{d\phi}{dt}}. \end{aligned} \quad (31)$$

In terms of the impact parameter b , $ds^2 = 0$ leads to the orbit equation on the equatorial plane as

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{A(r)D(r) + H^2(r)}{B(r)} \frac{D(r) - 2H(r)b - A(r)b^2}{[H(r) + A(r)b]^2}, \quad (32)$$

where we use Eq. (14). Let us introduce $u \equiv 1/r$ to rewrite the orbit equation as

$$\left(\frac{du}{d\phi}\right)^2 = F(u), \quad (33)$$

where $F(u)$ is

$$F(u) = \frac{u^4(AD + H^2)(D - 2Hb - Ab^2)}{B(H + Ab)^2}. \quad (34)$$

Finally, we examine the angles at the receiver and source positions. The unit tangent vector along the photon orbit in ${}^{(3)}M$ is e^i . On the equatorial plane, its components are obtained as

$$e^i = \frac{1}{\xi} \left(\frac{dr}{d\phi}, 0, 1 \right). \quad (35)$$

Here, ξ satisfies

$$\frac{1}{\xi} = \frac{A(r)[H(r) + A(r)b]}{A(r)D(r) + H^2(r)}, \quad (36)$$

which can be derived from $\gamma_{ij}e^ie^j = 1$ by using Eq. (32).

The unit radial vector in the equatorial plane is

$$R^i = \left(\frac{1}{\sqrt{\gamma_{rr}}}, 0, 0 \right), \quad (37)$$

where we choose the outgoing direction for a sign convention.

Therefore, we can define the angle measured from the outgoing radial direction by

$$\begin{aligned} \cos \Psi &\equiv \gamma_{ij}e^iR^j \\ &= \sqrt{\gamma_{rr}} \frac{A(r)[H(r) + A(r)b]}{A(r)D(r) + H^2(r)} \frac{dr}{d\phi}, \end{aligned} \quad (38)$$

where Eqs. (35), (36) and (37) are used. This can be rewritten as

$$\sin \Psi = \frac{H(r) + A(r)b}{\sqrt{A(r)D(r) + H^2(r)}}, \quad (39)$$

where we use Eq. (32). Note that $\sin \Psi$ by Eq. (39) is more convenient in practical calculations, because it needs only the local quantities, whereas $\cos \Psi$ by Eq. (38) needs the derivative as $dr/d\phi$.

D. Deflection angle of light

For the equatorial case in the axisymmetric spacetime, we define

$$\alpha \equiv \Psi_R - \Psi_S + \phi_{RS}. \quad (40)$$

This definition seems to rely on a choice of the angular coordinate ϕ . By using the Gauss-Bonnet theorem Eq. (8), this is rewritten as

$$\alpha = - \iint_{\infty \square \infty} K dS - \int_R^S \kappa_g d\ell, \quad (41)$$

where $d\ell$ is positive for the prograde motion of the photon and it is negative for the retrograde motion. Eq. (41) shows that α is coordinate-invariant also for the axisymmetric case.

Up to this point, equations for gravitational fields are not specified. Therefore, the above discussion and results are not limited within the theory of general relativity (GR) but they are applicable to a certain class of modified gravity theories if the light ray in the four-dimensional spacetime obeys the null geodesic.

IV. APPLICATION TO THE KERR LENS

A. Kerr spacetime and γ_{ij}

This section focuses on the Kerr spacetime as one of the most known examples with axisymmetry. The Boyer-Lindquist form of the Kerr metric is

$$ds^2 = - \left(1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4aMr \sin^2 \theta}{\Sigma} dt d\phi \\ + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2a^2 Mr \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2, \quad (42)$$

where Σ and Δ are denoted as

$$\Sigma \equiv r^2 + a^2 \cos^2 \theta, \quad (43)$$

$$\Delta \equiv r^2 - 2Mr + a^2. \quad (44)$$

By using Eqs. (16) and (17), one can see that γ_{ij} and β_i for the Kerr metric are given by

$$\gamma_{ij} dx^i dx^j = \frac{\Sigma^2}{\Delta(\Sigma - 2Mr)} dr^2 + \frac{\Sigma^2}{(\Sigma - 2Mr)} d\theta^2 \\ + \left(r^2 + a^2 + \frac{2a^2 Mr \sin^2 \theta}{(\Sigma - 2Mr)} \right) \frac{\Sigma \sin^2 \theta}{(\Sigma - 2Mr)} d\phi^2, \quad (45)$$

$$\beta_i dx^i = - \frac{2aMr \sin^2 \theta}{(\Sigma - 2Mr)} d\phi. \quad (46)$$

Note that γ_{ij} has no terms linear in the Kerr parameter a , because $g_{0i} \propto H$ enters γ_{ij} in a quadratic form through $g_{0i}g_{0j} \propto H^2$ as shown by Eq. (16).

In order to see what is κ_g for the present case, we employ the weak field and slow rotation approximations, for which M and a can be used as book-keeping parameters.

B. Path integral of κ_g

By substituting β_i by Eq. (46) into Eq. (28), we obtain

$$\kappa_g = - \frac{2aM}{r^2(r - 2M)} \left(\frac{1 - \frac{2M}{r} + \frac{a^2}{r^2}}{1 + \frac{a^2}{r^2} + \frac{2a^2 M}{r^3}} \right)^{1/2} \\ = - \frac{2aM}{r^3} + O(aM^2, a^3M), \quad (47)$$

where we use the weak field and slow rotation approximations in the last line.

The path integral of κ_g is computed as

$$\begin{aligned}
\int_R^S \kappa_g d\ell &= - \int_R^S \frac{2aM}{r^3} d\ell + O(aM^2, a^3M) \\
&= \frac{2aM}{b^2} \int_{\phi_R}^{\phi_S} \cos \vartheta d\vartheta + O(aM^2, a^3M) \\
&= \frac{2aM}{b^2} [\sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2}] + O(aM^2, a^3M),
\end{aligned} \tag{48}$$

where we assume the prograde case $d\ell > 0$ that the orbital angular momentum of the photon is aligned with the spin of the black hole and we use a linear approximation of the photon orbit as $r = b/\cos \vartheta + O(M, a)$ and $\ell = b \tan \vartheta + O(M, a)$ in the second line. Note that, in the retrograde case, the sign of $d\ell$ is negative and thus the magnitude of the above path integral remains the same but the sign is opposite.

C. ϕ_{RS} part

The integral of the angular coordinate ϕ becomes

$$\begin{aligned}
\phi_{RS} &= \int_S^R d\phi \\
&= 2 \int_0^{u_0} \frac{1}{\sqrt{F(u)}} du + \int_{u_S}^0 \frac{1}{\sqrt{F(u)}} du + \int_{u_R}^0 \frac{1}{\sqrt{F(u)}} du,
\end{aligned} \tag{49}$$

where we use the orbit equation given by Eq. (33), By substituting Eq. (34) into $F(u)$ in Eq. (49), we obtain

$$\begin{aligned}
\phi_{RS} &= \int_{u_S}^{u_0} \left(\frac{1}{\sqrt{u_0^2 - u^2}} + M \frac{u_0^3 - u^3}{(u_0^2 - u^2)^{3/2}} - 2aM \frac{u_0^3(u_0 - u)}{(u_0^2 - u^2)^{3/2}} \right) du \\
&\quad + \int_{u_R}^{u_0} \left(\frac{1}{\sqrt{u_0^2 - u^2}} + M \frac{u_0^3 - u^3}{(u_0^2 - u^2)^{3/2}} - 2aM \frac{u_0^3(u_0 - u)}{(u_0^2 - u^2)^{3/2}} \right) du \\
&\quad + O(M^2, a^2) \\
&= \left(\frac{\pi}{2} - \arcsin \left(\frac{u_S}{u_0} \right) + M \frac{(2u_0 + u_S)\sqrt{u_0^2 - u_S^2}}{u_0 + u_S} - 2aM \frac{u_0^3 \sqrt{u_0^2 - u_S^2}}{u_0^2 + u_0 u_S} \right) \\
&\quad + \left(\frac{\pi}{2} - \arcsin \left(\frac{u_R}{u_0} \right) + M \frac{(2u_0 + u_R)\sqrt{u_0^2 - u_R^2}}{u_0 + u_R} - 2aM \frac{u_0^3 \sqrt{u_0^2 - u_R^2}}{u_0^2 + u_0 u_R} \right) \\
&\quad + O(M^2, a^2),
\end{aligned} \tag{50}$$

where we assume the prograde case. For the retrograde case, the sign of the term linear in a becomes opposite.

Eq. (33) gives the relation between the impact parameter b and the inverse of the closest approach u_0 as $b = u_0^{-1} + M - 2aMu_0 + O(M^2, a^2)$ in the weak field and slow rotation approximations. By using this relation, aM part of ϕ_{RS} in Eq. (50) can be rewritten in terms of b as

$$-\frac{2aM}{b^2} \left[\frac{1}{\sqrt{1-b^2u_S^2}} + \frac{1}{\sqrt{1-b^2u_R^2}} \right]. \quad (51)$$

See Eq. (32) of Ref. [32] for M part of ϕ_{RS} .

D. Ψ parts

For the Kerr metric by Eq. (42), Eq. (39) becomes

$$\sin \Psi = \frac{b}{r} \times \frac{1 - \frac{2M}{r} + \frac{2aM}{br}}{\sqrt{1 - \frac{2M}{r} + \frac{a^2}{r^2}}}. \quad (52)$$

This is approximated as

$$\sin \Psi = \frac{b}{r} \left(1 - \frac{M}{r} + \frac{2aM}{br} \right) + O(M^2, aM^2). \quad (53)$$

By using this, we obtain

$$\begin{aligned} \Psi_R - \Psi_S &= \arcsin(bu_R) + \arcsin(bu_S) - \pi \\ &\quad - \frac{Mbu_R^2}{\sqrt{1-b^2u_R^2}} - \frac{Mbu_S^2}{\sqrt{1-b^2u_S^2}} \\ &\quad + \frac{2aMu_R^2}{\sqrt{1-b^2u_R^2}} + \frac{2aMu_S^2}{\sqrt{1-b^2u_S^2}} + O(M^2, aM^2). \end{aligned} \quad (54)$$

E. Deflection angle of light in Kerr spacetime

By substituting Eqs. (51) and (54) into Eq. (40), the deflection angle of light on the equatorial plane in the Kerr spacetime is obtained as

$$\begin{aligned} \alpha_{prog} &= \frac{2M}{b} \left(\sqrt{1-b^2u_S^2} + \sqrt{1-b^2u_R^2} \right) \\ &\quad - \frac{2aM}{b^2} \left(\sqrt{1-b^2u_R^2} + \sqrt{1-b^2u_S^2} \right) + O(M^2, aM^2), \end{aligned} \quad (55)$$

where we assume the prograde motion of light. For the retrograde case, it is

$$\begin{aligned}\alpha_{retro} = & \frac{2M}{b} \left(\sqrt{1 - b^2 u_S^2} + \sqrt{1 - b^2 u_R^2} \right) \\ & + \frac{2aM}{b^2} \left(\sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right) + O(M^2, aM^2).\end{aligned}\quad (56)$$

For both cases, we take the far limit as $u_R \rightarrow 0$ and $u_S \rightarrow 0$. Then, we obtain

$$\alpha_{\infty prog} \rightarrow \frac{4M}{b} - \frac{4aM}{b^2} + O(M^2, aM^2), \quad (57)$$

$$\alpha_{\infty retro} \rightarrow \frac{4M}{b} + \frac{4aM}{b^2} + O(M^2, aM^2), \quad (58)$$

which show that Eqs. (55) and (56) recover the asymptotic deflection angles that are known in literature [4, 34, 35].

F. Finite-distance corrections to the gravitomagnetic deflection angle of light

The above calculations discuss the deflection angle of light due to the rotation of the lens (its spin parameter a). In particular, we do not assume that the receiver and the source are located at the infinity. The finite-distance correction to the deflection angle of light, denoted as $\delta\alpha$, is the difference between the asymptotic deflection angle α_∞ and the deflection angle for the finite distance case. It is expressed as

$$\delta\alpha \equiv \alpha - \alpha_\infty. \quad (59)$$

Eqs. (55) and (56) suggest the magnitude of the finite-distance correction to the gravitomagnetic deflection angle by the spin as

$$\begin{aligned}|\delta\alpha_{GM}| & \sim O\left(\frac{aM}{r_S^2} + \frac{aM}{r_R^2}\right) \\ & \sim O\left(\frac{J}{r_S^2} + \frac{J}{r_R^2}\right),\end{aligned}\quad (60)$$

where $J \equiv aM$ is the spin angular momentum of the lens and the subscript GM denotes the gravitomagnetic part. As usual, we introduce the dimensionless spin parameter as $s \equiv a/M$. Hence, Eq. (60) is rewritten as

$$|\delta\alpha_{GM}| \sim O\left(s\left(\frac{M}{r_S}\right)^2 + s\left(\frac{M}{r_R}\right)^2\right). \quad (61)$$

This suggests that $\delta\alpha$ is comparable to the second post-Newtonian effect (multiplied by the dimensionless spin parameter). See also the next subsection.

Note that $\delta\alpha$ at the leading order in the approximations does not depend on the impact parameter b . In fact, $\delta\alpha$ depends much weakly on b .

G. Possible astronomical applications

We discuss possible astronomical applications. First, we consider the Sun, where we ignore its higher multipole moments. The spin angular momentum of the Sun J_\odot is $\sim 2 \times 10^{41} \text{ m}^2 \text{ kg s}^{-1}$ [49]. Thus, $GJ_\odot c^{-2} \sim 5 \times 10^5 \text{ m}^2$, which implies the dimensionless spin parameter as $s_\odot \sim 10^{-1}$.

We assume that an observer at the Earth sees the light bending by the solar mass, while the source is practically at the asymptotic region. If the light ray passes near the solar surface, Eq. (61) implies that the finite-distance correction to this case is of the order of

$$\begin{aligned} |\delta\alpha_{GM}| &\sim O\left(\frac{J}{r_R^2}\right) \\ &\sim 10^{-12} \text{ arcsec.} \times \left(\frac{J}{J_\odot}\right) \left(\frac{1\text{AU}}{r_R}\right)^2, \end{aligned} \quad (62)$$

where $4M_\odot/R_\odot \sim 1.75 \text{ arcsec.} \sim 10^{-5} \text{ rad.}$, and R_\odot denotes the solar radius. This correction is around a pico-arcsecond level and thus it is unlikely to be observed with present technology [50, 51].

Please see Figure 3 for numerical calculations of the finite-distance correction due to the receiver location. The numerical results are consistent with the above order-of-magnitude estimation. The figure suggests that the dependence of $\delta\alpha$ on the impact parameter b is very weak.

Next, we consider Sgr A* at the center of our Galaxy, which is expected as one of the most plausible candidates for the strong deflection of light. In this case, the receiver distance is much larger than the impact parameter of light, while a source star may be in the central region of our Galaxy.

For Sgr A*, Eq. (60) implies

$$|\delta\alpha_{GM}| \sim s \left(\frac{M}{r_S} \right)^2 \sim 10^{-7} \text{arcsec.} \times \left(\frac{s}{0.1} \right) \left(\frac{M}{4 \times 10^6 M_\odot} \right)^2 \left(\frac{0.1 \text{pc}}{r_S} \right)^2, \quad (63)$$

where we assume the mass of the central black hole as $M \sim 4 \times 10^6 M_\odot$. This correction around a sub-microarcsecond level is unlikely to be measured with present technology.

Please see Figure 4 for numerical calculations of the finite-distance correction due to the source location. The numerical results are consistent with the above order-of-magnitude estimation. The figure shows that the dependence on the impact parameter b is very weak.

H. Consistency of the present formulation

Before closing this section, let us check the consistency of the above formulation. The Gaussian curvature is related with the 2-dimensional Riemann tensor as [38]

$$K = \frac{{}^{(3)}R_{r\phi r\phi}}{\gamma} = \frac{1}{\sqrt{\gamma}} \left[\frac{\partial}{\partial \phi} \left(\frac{\sqrt{\gamma}}{\gamma_{rr}} {}^{(3)}\Gamma_{rr}^\phi \right) - \frac{\partial}{\partial r} \left(\frac{\sqrt{\gamma}}{\gamma_{rr}} {}^{(3)}\Gamma_{r\phi}^\phi \right) \right], \quad (64)$$

where ${}^{(3)}\Gamma_{jk}^i$ and ${}^{(3)}R_{abcd}$ are associated with γ_{ij} . For the Kerr case, it becomes

$$K = - \sqrt{\frac{A^3}{B(AD + H^2)}} \frac{\partial}{\partial r} \left[\frac{1}{2} \sqrt{\frac{A^3}{B(AD + H^2)}} \frac{\partial}{\partial r} \left(\frac{AD + H^2}{A^2} \right) \right] = - \frac{2M}{r^3} + O(M^2, a^2), \quad (65)$$

where we use the weak field and slow rotation approximations in the last line. Note that K has no terms linear in a . This is because γ_{ij} has no terms linear a as already mentioned.

In order to compute the surface integral of the Gaussian curvature in the Gauss-Bonnet theorem, we need know the integration domain, especially the photon orbit $S \rightarrow R$ for the present case. By straightforward calculations, the iterative solution of Eq. (33) for the Kerr case in the weak field and slow rotation approximations is obtained as

$$u = \frac{1}{b} \sin \phi + \frac{M}{b^2} (1 + \cos^2 \phi) - \frac{2aM}{b^3} + O(M^2, aM^2). \quad (66)$$

By using this, the surface integral of the Gaussian curvature is computed as

$$\begin{aligned}
\iint_{\infty_R \square \infty_S} K dS &= \int_{\infty}^{r_{OE}} dr \int_{\phi_S}^{\phi_R} d\phi \frac{2M}{r^2} + O(M^2, aM^2) \\
&= 2M \int_{\phi_S}^{\phi_R} d\phi \int_0^{\frac{1}{b} \sin \phi + \frac{M}{b^2}(1+\cos^2 \phi) - \frac{2aM}{b^3}} du + O(M^2, aM^2) \\
&= 2M \int_{\phi_S}^{\phi_R} d\phi \left[u \right]_{u=0}^{\frac{1}{b} \sin \phi + \frac{M}{b^2}(1+\cos^2 \phi) - \frac{2aM}{b^3}} + O(M^2, aM^2) \\
&= \frac{M}{b} \int_{\phi_S}^{\phi_R} d\phi \sin \phi + O(M^2, aM^2) \\
&= \frac{2M}{b} \left[\sqrt{1 - b^2 u_S^2} + \sqrt{1 - b^2 u_R^2} \right] + O(M^2, aM^2). \tag{67}
\end{aligned}$$

By combining Eqs. (48) and (67), we obtain

$$\begin{aligned}
- \iint_{\infty_R \square \infty_S} K dS - \int_R^S \kappa_g d\ell &= \frac{2M}{b} \left(\sqrt{1 - b^2 u_S^2} + \sqrt{1 - b^2 u_R^2} \right) \\
&\quad - \frac{2aM}{b^2} \left(\sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right) \\
&\quad + O(M^2, aM^2). \tag{68}
\end{aligned}$$

This equals to the right-hand side of Eq. (55). This means that the present approach is consistent with the Gauss-Bonnet theorem.

V. CONCLUSION

By using the Gauss-Bonnet theorem in differential geometry, we discussed a possible extension of the method of calculating the bending angle of light to stationary, axisymmetric and asymptotically flat spacetimes. We introduced a spatial metric γ_{ij} to define the bending angle of light, which was shown to be coordinate-invariant.

We considered the light rays on the equatorial plane in the axisymmetric spacetime. We showed that the geodesic curvature of the photon orbit with γ_{ij} can be nonzero in gravitomagnetism, even though the light ray in the four-dimensional spacetime follows the null geodesic. Finally, we considered Kerr spacetime in order to examine how the bending angle of light is computed by the present method. We made an order-of-magnitude estimate of the finite-distance corrections for two possible astronomical cases; (1) the Sun and (2) the Sgr A*. The results suggest that the finite-distance corrections due to gravitomagnetism are unlikely to be observed with present technology.

However, our analysis on possible astronomical observations in this paper is limited within the Kerr model. It might be interesting to examine the gravitomagnetic bending of light by using other axisymmetric spacetimes in GR or in a specific theory of modified gravity. A further study along this direction is left for future.

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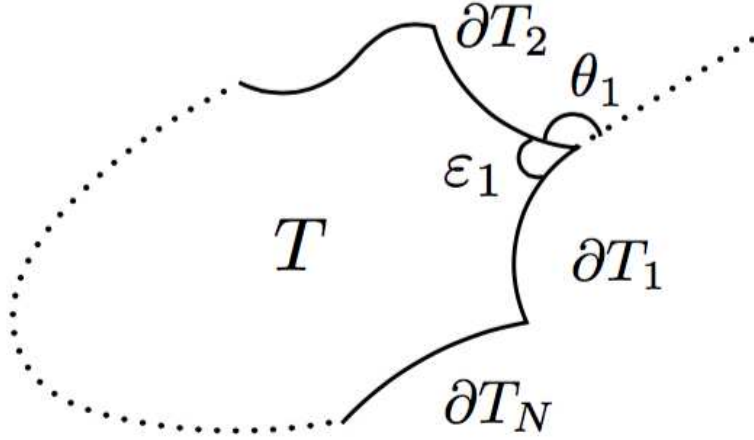


FIG. 1: Schematic figure for the Gauss-Bonnet theorem.

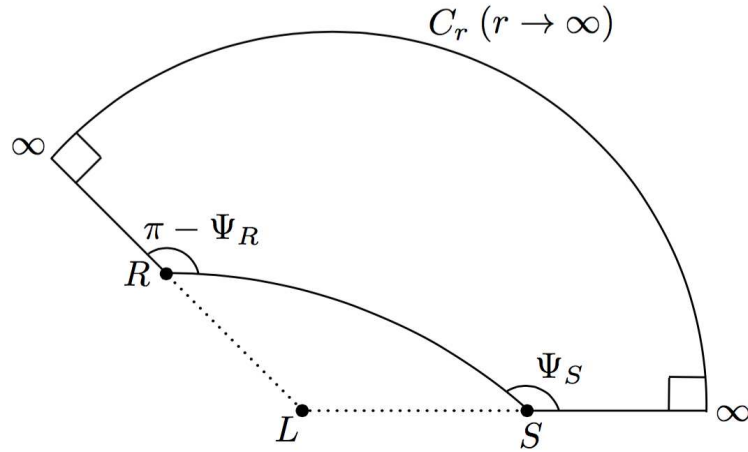


FIG. 2: Quadrilateral ${}^{\infty}\square_S^{\infty}$ embedded in a curved space. Note that the inner angle at the vertex R is $\pi - \Psi_R$.

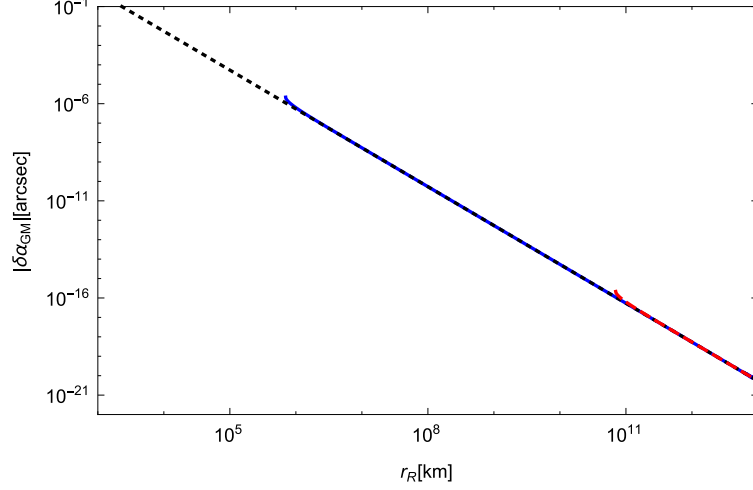


FIG. 3: $\delta\alpha_{GM}$ for the Sun. The vertical axis denotes the finite-distance correction to the gravitomagnetic deflection angle of light and the horizontal axis denotes the receiver distance r_R . The solid curve (blue in color) and dashed one (red in color) correspond to $b = R_\odot$ and $b = 10R_\odot$, respectively. The dotted line (black in color) denotes the leading term of $\delta\alpha_{GM}$ given by Eq. (60). The overlap between these curves suggest that the dependence of $\delta\alpha_{GM}$ on the impact parameter b is very weak.

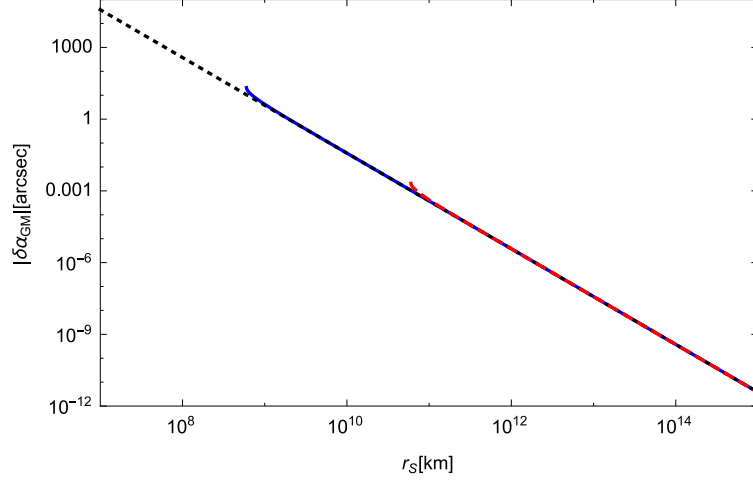


FIG. 4: $\delta\alpha_{GM}$ for the Sgr A*. The vertical axis denotes the finite-distance correction to the deflection angle of light and the horizontal axis denotes the source distance r_S . The solid curve (blue in color) and dashed one (red in color) correspond to $b = 10^2 M$ and $b = 10^4 M$, respectively. The dotted line (yellow in color) denotes the leading term of $\delta\alpha_{GM}$ given by Eq. (60). The overlap between these plots suggest that $\delta\alpha_{GM}$ depends faintly on the impact parameter b .